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Reproducing Kernels for Elliptic Systems*

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1. INTRODUCTION

The kernel function method as presented by Bergman and Schiffer has had an important impact on approximation theory and the numerical treatment of elliptic differential equations [1–4]. With this as motivation, we have developed a kernel function theory for elliptic systems of differential equations. Here we treat the case of the self-adjoint system:

$$\Delta U = C(x, y)U \tag{1.1}$$

where U and C are complex valued $n \times n$ matrices and Δ is the Laplace operator in two dimensions:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

However, many of our results can be extended to more general self-adjoint systems, and also to higher-order and higher-dimensional cases.

Equation (1.1) will be considered on a bounded regular region D in the Euclidean plane and for simplicity $\dot{D}$, the boundary of D, will be assumed to be analytic. In $\bar{D}$ the matrix C will be assumed to be positive definite and Hermitian: $C = C^*$, where C^* is the conjugate-transpose of C; also, C will be assumed to belong to $\mathscr{C}'(\bar{D})$.

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It is interesting that many of the results of the scalar case can be shown to have analogues in the matrix case. To make these similarities more obvious we have been careful to choose notations that make the results appear to be formally identical, whenever possible. Because of the noncommutivity of the matrices involved, we stress that the order in which terms appear in our formulas is essential.

2. Fundamental Matrices and Green-Dirichlet Identities

By $\Omega \equiv \Omega(\bar{D})$, we denote the set of all $n \times n$ $(n \geqslant 1)$ matrices, whose entries are complex valued functions of class $\mathscr{C}^2(\bar{D})$. For any V, W in Ω define

$$E\{V, W\} \equiv \iint_D [V_x{}^* W_x + V_y{}^* W_y + V^* C W] \, dx \, dy. \tag{2.1}$$

It follows that $E^*\{V, W\} = E\{W, V\}$, and that E^* is sesquilinear. The quadratic mapping $E\{V, V\}$, abbreviated $E\{V\}$, is a positive semidefinite Hermitian matrix. We observe that $E\{V, WA\} = E\{V, W\}A$ for every constant $(n \times n)$ matrix A. Hence we have an *inner product* structure as in Hilbert modules [5, 6]. The matrix norm used for Hilbert modules, however, seems not to be as convenient for our purposes as the following

$$\| V \| \equiv \sup_{|\xi|=1} | \xi^* V \xi | \tag{2.2}$$

where $\xi \in C^n$. One may show that the *Schwarz inequality*

$$\| E\{V, W\} \| \leqslant \| E\{V\} \|^{1/2} \| E\{W\} \|^{1/2}, \tag{2.3}$$

and the triangle inequality,

$$\| E\{V + W\} \|^{1/2} \leqslant \| E\{V\} \|^{1/2} + \| E\{W\} \|^{1/2}, \tag{2.4}$$

are valid. Moreover, $\| E\{V\} \|^{1/2}$ defines a norm on Ω.

For each V, W in Ω we have the first and second Green's identities for the formally self-adjoint operator $L \equiv \Delta - C$, i.e.,

$$\begin{aligned} E\{V, W\} &= - \int_{\dot{D}} V^* (\partial W / \partial \nu) \, ds - \iint_D V^* L[W] \, dx \, dy \\ &= - \int_{\dot{D}} (\partial V^* / \partial \nu) \, W \, ds - \iint_D (L[V])^* \, W \, dx \, dy \end{aligned} \tag{2.5}$$

and

$$\iint_D [V^* L[W] - (L[V])^* \, W] \, dx \, dy = \int_{\dot{D}} [(\partial V^* / \partial \nu) \, W - V^* (\partial W / \partial \nu)] \, ds. \tag{2.6}$$

Differentiation in the normal direction is with respect to the inner normal (unit) vector ν.

By a fundamental matrix of L with pole Q, we mean a matrix $S = S(P, Q)$ of the form

$$S(P, Q) = [I/(2\pi)] \log(1/r) + s(P, Q) \tag{2.7}$$

which is a solution for the equation

$$L_p[S](P, Q) = -\delta(P - Q)I.$$

Here I is the $(n \times n)$ identity matrix and δ is the Dirac delta. The matrix S is of class $\mathscr{C}^2(\bar{D} \times \bar{D})$ except for $P = Q$ where it is $\mathscr{C}'$. The existence of S is assured by integral equations methods.

From (2.5) we obtain for V in Ω the representation formula

$$V(Q) = \int_{\dot{D}} (\partial S^*/\partial \nu_p)(P, Q)\, V(P)\, ds_p + E\{S(P, Q), V(P)\}. \tag{2.8}$$

We denote by Σ that subset of Ω consisting of the classical solutions of (1.1). For any U in Σ the identity (2.6) yields

$$U(Q) = \int_{\dot{D}} [(\partial S^*/\partial \nu_p)(P, Q)\, U(P) - S^*(P, Q)(\partial U(P)/\partial \nu_p)]\, ds_p . \tag{2.9}$$

In what follows, we will assume that the fundamental matrix $S(P, Q)$ will remain fixed and introduce the Green matrix $G(P, Q)$, Neumann matrix $N(P, Q)$, and Robin matrix $R(P, Q)$ for L with respect to the region D. Each of these matrices is a particular fundamental matrix satisfying conditions on $\dot{D}$. For P on $\dot{D}$ and Q in D and given continuous matrix Λ on $\dot{D}$

$$G(P, Q) = 0, \ (\partial N/\partial \nu_p)(P, Q) = 0 \quad \text{and} \quad (\partial/\partial \nu_p)\, R(P, Q) = \Lambda^*(P)\, R(P, Q).$$

Here Λ is also assumed to be positive-definite on $\dot{D}$. Like S, each of these fundamental matrices belongs to $\mathscr{C}^2(\bar{D} \times \bar{D})$ except for $P = Q$. The usual representation formulae for the first, second, and third boundary value problems are given, respectively, by

$$U(Q) = \int_{\dot{D}} (\partial G^*/\partial \nu_p)(P, Q)\, U(P)\, ds_p , \tag{2.10}$$

$$U(Q) = -\int_{\dot{D}} N^*(P, Q)(\partial U/\partial \nu_p)(P)\, ds_p , \tag{2.11}$$

and

$$U(Q) = -\int_{\dot{D}} R^*(P, Q)[(\partial U/\partial \nu_p)(P) - \Lambda(P)\, U(P)]\, ds_p . \tag{2.12}$$

Remark. The uniqueness of the Dirichlet, Neumann, and Robin problems follows, as usual, from Green's Identity (2.5).

The compensating parts of G, N, and R are the matrices defined by

$$g(P, Q) \equiv G(P, Q) - S(P, Q)$$
$$n(P, Q) \equiv N(P, Q) - S(P, Q)$$
and
$$r(P, Q) \equiv R(P, Q) - S(P, Q).$$

Each of these matrices belongs to $\mathscr{C}^2(\bar{D} \times \bar{D})$ except for both P and Q on $\dot{D}$ and $P = Q$. The symmetry properties $G^*(P, Q) = G(Q, P)$ and $N^*(P, Q) = N(Q, P)$ are a consequence, as in the scalar case, of (2.6).

From (2.8), we have the following reproducing formulas for the spaces Ω, and Ω°, i.e.,

$$V(Q) = E\{N(P, Q), V(P)\}, \qquad V \in \Omega, \tag{2.13}$$

and

$$V(Q) = E\{G(P, Q), V(P)\}, \qquad V \in \Omega^\circ, \tag{2.14}$$

where Ω° is that subset of Ω whose matrix functions vanish on $\dot{D}$. On the other hand,

$$E\{G(P, Q), U(P)\} = 0, \qquad U \in \Sigma \tag{2.15}$$

may be obtained from (2.5).

Henceforth, we will assume that the fixed fundamental matrix has the symmetry property

$$S^*(P, Q) = S(Q, P). \tag{2.16}$$

Since G and N have this property, there is no loss of generality in making such an assumption. It follows that g and n must also have this symmetry property.

3. The Matrix Kernel and the Dirichlet Identities

The matrix kernel K is defined by

$$K(P, Q) \equiv N(P, Q) - G(P, Q), \tag{3.1}$$

from which the reproducing property

$$U(Q) = E\{K(P, Q), U(P)\}, \qquad U \in \Sigma, \tag{3.2}$$

follows directly via (2.13) and (2.15). As in the scalar case, K belongs to $\mathscr{C}^2(\bar{D} \times \bar{D})$ except for P and Q both on $\dot{D}$ and $P = Q$.

The symmetry property of K, $K^*(P, Q) = K(Q, P)$, is an immediate consequence of its definition.

If a matrix $K(P, Q)$ having the reproducing property (3.2) is known, then G may be obtained from

$$G(R, Q) = S(R, Q) - E\{K(P, R), S(P, Q)\},$$

which then also yields N by (3.1).

The matrix kernel also permits estimates for elements of Σ directly from (3.2), i.e.,

$$\begin{aligned} \| U(Q)\|^2 &\leqslant \| E\{K(P, Q)\}\| \, \| E\{U\}\| \\ &= \| K(Q, Q)\| \, \| E\{U\}\|. \end{aligned} \tag{3.3}$$

This inequality is sharp, since equality occurs for $U(P) = K(P, Q)$. This observation yields the following characterization of K as an extremal function for the minimum problem

$$\min(\| E\{U\}\| / \| U(Q)\|^2) = 1/\| K(Q, Q)\|, \tag{3.4}$$

where U is subject to the restriction that it satisfy our partial differential equation.

The inequality (3.3) applied to $U(P) = K(P, T)$ yields the interesting estimate

$$\| K(Q, T)\|^2 \leqslant \| K(T, T)\| \, \| K(Q, Q)\|. \tag{3.5}$$

Following Bergman and Schiffer [1, p. 298] it will be convenient to introduce the geometric quantity

$$I(P, Q) \equiv \int_{\dot{D}} (\partial S^*/\partial \nu_T)(T, P)\, S(T, Q)\, ds_T\,, \tag{3.6}$$

which has the symmetry property since

$$\begin{aligned} I^*(P, Q) - I(Q, P) &= \int_{\dot{D}} \left[S^*(T, Q) \frac{\partial S(T, P)}{\partial \nu_T} - \frac{\partial S^*(T, Q)}{\partial \nu_T} S(T, P) \right] ds_T \\ &= S^*(P, Q) - S(Q, P) = 0 \end{aligned} \tag{3.7}$$

as follows from Green's second identity (2.6).

There are matrix analogues of the Dirichlet identities also, which we list below,

$$E\{ g(T, P), g(T, Q)\} = -g(P, Q) - I(P, Q) \tag{3.8}$$

$$E\{ g(T, P), n(T, Q)\} = -I(P, Q) = E\{n(T, P), g(T, Q)\} \tag{3.9}$$

and

$$E\{n(T, P), n(T, Q)\} = n(P, Q) - I(P, Q) \tag{3.10}$$

so that

$$E\{l(T, P), l(T, Q)\} = K(P, Q) - 4I(P, Q) \tag{3.11}$$

where the l kernel is defined by

$$l(P, Q) \equiv n(P, Q) + g(P, Q). \tag{3.12}$$

By an adaptation of the proof for the scalar case in [1] one can show that l belongs to $\mathscr{C}'(\bar{D} \times \bar{D})$; moreover, the mixed second derivatives $l_{x\xi}$, etc., are continuous for P distinct from Q and $(\log r_{PQ})^{-1}\, l_{x\xi}$, etc., remain bounded for P tending to Q.

4. An Integral Equation for the Matrix Kernel

We now characterize the matrix kernel K as the solution of an integral equation and thereby obtain useful series developments for K.

Since, for fixed Q, $K(P, Q) - 4I(P, Q)$ is an element of Σ, the reproducing property (3.2) together with (2.5) implies

$$K(Q, P) - 4I(Q, P) = -\int_{\dot{D}} \mathscr{K}^*(Q, T)\, K(T, P)\, ds_T .$$

Here $\mathscr{K}$ is defined by

$$\mathscr{K}(Q, T) \equiv (\partial/\partial\nu_T)[K(T, Q) - 4I(T, Q)]$$

and is seen to be continuous on $\dot{D} \times \dot{D}$ due to the $\mathscr{C}'$ behavior of l and (3.11). We are led therefore to consider the homogeneous equation

$$\Phi(Q) = -\lambda \int_{\dot{D}} \mathscr{K}^*(Q, T)\, \Phi(T)\, ds_T \tag{4.1}$$

and investigate its eigenvalues.

The following lemmas are easily proven:

Lemma. *The eigenvalues of* (4.1) *are precisely those of the eigenvalue problem*

$$U(Q) = \lambda E\{M(P, Q), U(P)\} \tag{4.2}$$

where

$$M(P, Q) \equiv K(P, Q) - 4I(P, Q).$$

LEMMA. *The eigenvalues of* (4.2) *are precisely the squares of those of the eigenvalue problem*

$$V(Q) = \mu E\{l(P, Q), V(P)\}. \tag{4.3}$$

The proofs depend on the easily proven identity for elements of Ω:

$$E\{E\{X(T, P), Y(T)\}, Z(P)\} = E\{Y(T), E\{X^*(T, P), Z(P)\}\}. \tag{4.4}$$

In the next section we will prove:

LEMMA. *The eigenvalues of* (4.3) *are real.*

If we assume, as in the scalar case [1], that $S(P, Q) = G_1(P, Q)$ or $N_1(P, Q)$ for domain D_1 containing D, then it is easy to show that the mapping $\mathbf{T}$ on Σ defined by

$$\mathbf{T}[U](P) \equiv E\{l(R, P), U(R)\} \tag{4.5}$$

is norm-decreasing:

$$\| E\{T[U]\}\| \leqslant \| E\{U\}\|$$

with equality only for $U \equiv 0$. Hence, under our additional assumption about S, we have immediately:

LEMMA. *The eigenvalues of* (4.3) *satisfy*

$$\mu < -1 \qquad or \qquad \mu > 1.$$

As a result of these lemmas we see that λ in (4.1) satisfies $\lambda^2 > 1$ and the usual iterative scheme for solving the corresponding inhomogeneous equation

$$\Phi(P) = F(P) - \lambda \int_{\dot{D}} \mathscr{K}^*(P, T)\, \Phi(T)\, ds_T$$

leads for sufficiently small $|\lambda|$ to the series representation

$$\Phi(P) = F(P) + \sum_{\nu=1}^{\infty} (-\lambda)^\nu \int_{\dot{D}} \mathscr{K}_\nu^*(P, T) F(T)\, ds_T$$

as well as the representation

$$\Phi(P) = F(P) + \int_{\dot{D}} \mathscr{K}^{-1}(P, T, \lambda)\, F(T)\, ds_T \tag{4.6}$$

in terms of the resolvent kernel

$$\mathscr{K}^{-1}(P, T, \lambda) \equiv \sum_{\nu=1}^{\infty} (-\lambda)^\nu\, \mathscr{K}_\nu^*(P, T).$$

The νth iterated kernel $\mathscr{K}_\nu$ is defined by

$$\mathscr{K}_\nu^*(P, Q) \equiv \int_{\dot{D}} \mathscr{K}^*(P, T) \mathscr{K}_{\nu-1}^*(T, Q)\, ds_T$$

for $\nu \geqslant 2$ with

$$\mathscr{K}_1(T, Q) \equiv \mathscr{K}(T, Q).$$

As in the scalar case this allows one to obtain a series representation for K in terms of

$$M_N(P, Q) \equiv E\{M(T, P), M_{N-1}(T, Q)\}$$

where

$$M(T, P) \equiv M_1(T, P) \equiv K(T, P) - 4I(T, P).$$

Indeed, since

$$(-1)^{n-1}(\partial M_N(P, Q)/\partial \nu_p) = K_N(Q, P)$$

putting $\Phi(Q) = K(Q, P)$ and $F(Q) = 4I(Q, P)$ in the representation (4.6) yields

$$K(Q, P) = 4I(Q, P) + \int_{\dot{D}} \mathscr{K}^{-1}(Q, T)\, 4I(T, P)\, ds_T$$
$$= \sum_{\nu=0}^{\infty} E\{M_\nu(T, Q), 4I(T, P)\} \tag{4.7}$$

where $M_0(T, Q) \equiv K(T, Q)$. The expression (4.7) has the defect that K is contained on the right side. To remedy this, introduce the purely geometric quantities

$$i_1(P, Q) \equiv 4I(P, Q)$$
$$i_\nu(P, Q) \equiv E\{i_{\nu-1}^*(P, T), 4I(T, Q)\}, \qquad \nu \geqslant 2$$
$$= E\{4I(T, P), i_{\nu-1}(T, Q)\}$$

which satisfy

$$M_N(P, Q) = \sum_{\nu=0}^{N} (-1)^\nu \binom{N}{\nu} i_\nu(P, Q).$$

In terms of the i_ν one obtains the important representation

$$K(P, Q) = \sum_{\rho=0}^{\infty} \sum_{\nu=0}^{\rho} (-1)^\nu \binom{\rho}{\nu} i_{\nu+1}(P, Q) \tag{4.8}$$

which involves only integrals of the fundamental matrix S.

5. The Eigenvalue Problem

In this section we examine more closely the eigenvalue problem (4.3) and obtain Fourier expansions in terms of the corresponding matrices. The principal result is the elegant representation (5.9) for the kernel matrix.

To study (4.3) we introduce on Σ the following inner product

$$[U, V] \equiv \operatorname{trace} E\{V, U\}. \tag{5.1}$$

Let $\hat{\Sigma}$ be the completion of Σ with respect to $[\ ,\ ]$. The operator $\mathbf{T}$ involved in (4.3) and defined for elements U of Σ by (4.5) is linear, bounded, and symmetric, and it has a continuous extension to a compact self-adjoint operator $\hat{\mathbf{T}}$ on $\hat{\Sigma}$. Hence, $\hat{\mathbf{T}}$ has a countable set of real eigenvalues λ_ν ($|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$) with corresponding orthonormal eigenmatrices $\tilde{V}_\nu$:

$$\lambda_\nu \mathbf{T}[\tilde{V}_\nu] = \tilde{V}_\nu, \qquad [\tilde{V}_\mu, \tilde{V}_\nu] = \delta_{\mu\nu}. \tag{5.2}$$

To these $\tilde{V}_\nu$ we adjoin a countable orthonormal basis of elements of Σ for the null space of $\hat{\mathbf{T}}$ (the eigenmatrices corresponding to $\lambda_\nu = \infty$) to obtain a complete orthonormal system $\{V_\nu\}$ for $\hat{\Sigma}$. For any U in $\hat{\Sigma}$, one has the Fourier expansion

$$U = \sum_{\nu=1}^{\infty} \alpha_\nu V_\nu, \qquad \alpha_\nu \equiv [U, V_\nu] \tag{5.3}$$

convergent in the sense of $\hat{\Sigma}$. In particular,

$$K(P, Q) = \sum_{\nu=1}^{\infty} \alpha_\nu(Q)\, V_\nu(P). \tag{5.4}$$

The series (5.3) and (5.4) also converge in the sense of $\| E\{\ \}\|^{1/2}$ since this norm and $[\ ,\ ]^{1/2}$ are equivalent. However, to obtain expansions more attuned to the sesquilinear mapping E^* we renormalize in a different way. First, using (4.4), it is easy to see that the V_ν belonging to distinct eigenvalues (including $\lambda_\nu = \infty$) are orthogonal in the sense that

$$E\{V_\nu, V_\mu\} = 0. \tag{5.5}$$

The renormalization then proceeds via the following lemmas. As before, all matrices are $(n \times n)$ and I denotes the identity matrix.

Lemma 5.1. *If V is an eigenmatrix of $\mathbf{T}$ such that V is nonsingular at some point of D, then there exists a constant positive-definite Hermitian matrix R such that*

$$E\{VR^{-1}, VR^{-1}\} = I.$$

Proof. Merely choose R to be the unique positive-definite Hermitian square root of $E\{V\}$.

Remark. Note that VR^{-1} is an eigenmatrix of $\mathbf{T}$ belonging to the same eigenvalue as V. The same is true of VR_1 in the following lemma.

LEMMA 5.2. *If V is any eigenmatrix of $\mathbf{T}$, then there exists a nonsingular constant matrix R_1 such that*

$$E\{VR_1, VR_1\} = \tilde{I},$$

where $\tilde{I}$ is a nonzero diagonal idempotent matrix.

Proof. Since $E\{V\}$ is Hermitian and positive semidefinite, there exists a constant unitary A such that

$$E\{V\} = A^* \left(\sum_{j=1}^{n} k_j I_j \right) A$$

where $k_j \geqslant 0$ with at least one $k_j \neq 0$ and the idempotent matrices I_j have zero entries except for the entry 1 in jth diagonal position. Now let

$$\begin{aligned} \tilde{k}_j &= k_j^{-1/2}, \qquad \text{if} \quad k_j > 0 \\ &= 1, \qquad\quad \text{if} \quad k_j = 0 \end{aligned}$$

where the positive square root is chosen. Then $R_1 = A^* \sum_{j=1}^{n} \tilde{k}_j I_j$ satisfies all the requirements and the proof is complete.

Remark. The construction in this lemma is like that in [5, Lemma 2.1]. Now a Gram–Schmidt process will be carried out.

LEMMA 5.3. *For any eigenvalue $\lambda_\nu \neq \infty$ the corresponding eigenspace $\mathscr{E}_\nu$ is spanned by the matrix linear combinations*

$$\sum_{j=1}^{m} W_j(P) A_j \tag{5.6}$$

where the W_j are eigenmatrices of $\mathbf{T}$ belonging to λ_ν,

$$E\{W_i, W_j\} = 0, \qquad i \neq j \tag{5.7}$$

and

$$E\{W_j\} = \tilde{I}_j \tag{5.8}$$

where $\tilde{I}_j$ is a nonzero diagonal idempotent matrix.

Proof. Construct $(n \times n)$ matrices $\tilde{W}_1, \ldots, \tilde{W}_m$ $(m = m(\nu) \geqslant 1)$ from the linearly independent columns of all the V_μ in $\mathscr{E}_\nu$, using each column once and only once but filling in zero columns in $\tilde{W}_m$, if necessary. It is easy to see that:

(i) Each $\tilde{W}_j$ is an eigenmatrix of $\mathbf{T}$ belonging to the eigenvalue λ_ν.

(ii) Except possibly $\tilde{W}_m$ each $\tilde{W}_j$ is nonsingular.

(iii) The matrix linear combinations $\sum_{j=1}^{m} \tilde{W}_j(P)\, \tilde{A}_j$ span $\mathscr{E}_\nu$.

By the previous lemmas we may assume, without loss of generality, that $E\{\tilde{W}_m\} = \tilde{I}_m$ and $E\{\tilde{W}_j\} = I$ for $j < m$. Putting $W_1 = \tilde{W}_1$ and proceeding recursively

$$W_{j+1} = \tilde{W}_{j+1} - \sum_{l=1}^{j} W_l E\{W_l, \tilde{W}_{j+1}\}$$

we obtain the desired orthogonality property (5.7). Using Lemma 5.2, it may be assumed that the normalization condition (5.8) is satisfied by the W_j. Finally, from the definition of W_j in terms of $\tilde{W}_i$ the spanning property (5.6) is assured, completing the proof.

Remark. The eigenspace for $\lambda_\nu = \infty$ is also covered in the last lemma. If the null space of $\mathbf{T}$ is finite dimensional then the Lemma stands without alteration; otherwise, let $m = \infty$ in (5.6) and carry out the same construction on the W_j.

As a result of these considerations we are assured that there exists a set $\{U_\nu\}$ of eigenmatrices of $\mathbf{T}$ (including $\lambda_\nu = \infty$) with the orthonormal properties

$$E\{U_\nu, U_\mu\} = 0, \qquad \mu \neq \nu$$

and

$$E\{U_\nu, U_\nu\} = I_\nu$$

where I_ν is a nonzero diagonal idempotent matrix. The $\{U_\nu\}$ are complete in $\hat{\Sigma}$: Every $U \in \hat{\Sigma}$ has a representation with constant matrix coefficients

$$U = \sum_{\nu=1}^{\infty} U_\nu A_\nu, \qquad I_\nu A_\nu = E\{U_\nu, U\}$$

convergent with respect to $\| E\{\ \}\|^{1/2}$. In particular, from (5.4),

$$K(P, Q) = \sum_{\nu=1}^{\infty} U_\nu(P)\, A_\nu(Q)$$

with $I_\nu A_\nu(Q) = U_\nu{}^*(Q)$ so that $U_\nu(P)\, A_\nu(Q) = A_\nu{}^*(Q)\, I_\nu A_\nu(Q) = A_\nu{}^*(P)\, I_\nu I_\nu A_\nu(Q) = U_\nu(P)\, U_\nu{}^*(Q)$ and thus

$$K(P, Q) = \sum_{\nu=1}^{\infty} U_\nu(P)\, U_\nu{}^*(Q). \tag{5.9}$$

Observing that $U_\nu(P)\, U_\nu{}^*(Q) = U_\nu(P)\, I_\nu U_\nu{}^*(Q)$, we find in a similar way

$$I(P, Q) = \sum_{\nu=1}^{\infty} (U_\nu(P)\, U_\nu{}^*(Q)/\lambda_\nu) \tag{5.10}$$

and so by iteration

$$M(P, Q) = \sum_{\nu=1}^{\infty} (U_\nu(P)\, U_\nu{}^*(Q)/\lambda_\nu{}^2). \tag{5.11}$$

In view of the estimate (3.3), the series (5.9)–(5.11) converges uniformly in compact subsets of D. Moreover, series (5.11) converges uniformly on $\bar{D} \times \bar{D}$. For $P = Q$ this follows from the continuity on $M(Q, Q)$ on $\bar{D}$ and the positive definiteness of $U_\nu(Q)\, U_\nu{}^*(Q)\, \lambda_\nu^{-2}$ via Dini's theorem on uniform convergence. For general P, Q the result follows from the Cauchy inequality for infinite sums.

It is now a simple matter to follow the scalar case [1] to obtain the useful estimates

$$\| K(P, Q) - K_N(P, Q)\| \leqslant (1/\lambda_1^{2N}) \| M(P, P)\|^{1/2} \| M(Q, Q)\|^{1/2} \tag{5.12}$$

and

$$\| K(P, Q) - K_N(P, Q)\| \leqslant \{4/[\lambda_1^{2N}(\lambda_1{}^2 - 1)]\} \| I(P, P)\|^{1/2} \| I(Q, Q)\|^{1/2}$$

where K_N is the nth partial sum for K in (4.8):

$$K_N(P, Q) = \sum_{\rho=0}^{N} \sum_{\nu=0}^{\rho} (-1)^\nu \binom{\rho}{\nu} i_{\nu+1}(P, Q). \tag{5.13}$$

These estimates permit effective estimates for the error in the approximation of solutions of boundary value problems for (1.1) and related nonlinear problems.

The authors pursue this idea in a paper [7] concerning the semilinear matrix equation

$$\Delta U = f(x, y, U, U_x, U_y). \tag{5.14}$$

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